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A CHIP METHOD FOR LINEAR DISCRETE OPTIMIZATION PROBLEMS

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Abstract

In this paper, we propose a CHIP method for linear discrete optimization problems. Under conditions commonly used in the literature, a smooth path will

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be proven to exist. This will give a constructive proof of existence of solution and lead to an implementable globally convergent algorithm.

1. Introduction

In this paper, we consider the following linear discrete optimization problems:

$$\min_{x \in \mathbb{R}^n} f(x)$$

$$\text{s.t.} \quad Ax = b,$$

$$x \in \{0, 1\}^n, \tag{1.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^1$, $e = (1, \dots, 1)^T$ is a vector of dimension n.

Since Kellogg et al. (see [1]) and Smale (see [4]) proposed the notable homotopy method, this method has become a powerful tool in dealing with various nonlinear problems, for example, zeros or fixed points of maps. However, the homotopy method has seldom been touched in constrained optimization until 1988, Megiddo and Kojima et al. discovered the Karmarkar interior point method for linear programming was a kind of path-following method. From then on, the interior path-following methods for mathematical programming have become an active research subject, furthermore, it was extended to convex nonlinear programming problems recently.

Recently, a combined homotopy interior point method (CHIP method) for nonlinear programming problems was presented in [2]. In [3], for convex programming problems, compared with the interior path-following methods, the authors obtained the global convergence results without assuming the logarithmic barrier function to be strictly convex and the solution set to be bounded. This shows that the CHIP method can solve the problem that interior path-following methods cannot solve. In [5], by taking a piecewise technique, under commonly used conditions, polynomiality of the CHIP method is given, which shows that the efficiency of CHIP method is also very well. Up to now, the CHIP method has been applied to various areas such as fixed point problems, variational inequalities, multiobjective programming problems and so on.

However, there has no result on the CHIP method for solving problem (1.1) till now. In this paper, we attempt to complete this work. Under conditions commonly used in the literature, a smooth path from a given interior point to a solution of problem (1.1) will be proven to exist. This will give a constructive proof of existence of solution and lead to an implementable globally convergent algorithm to problem (1.1).

Throughout this paper, let $\Omega = \{x \in \mathbb{R}^n : 0 \leq x \leq e\}$, $\Omega^0 = \{x \in \mathbb{R}^n : 0 < x < e\}$, and $\partial\Omega = \Omega \setminus \Omega^0$ be the boundary set of Ω. In addition, denote the nonnegative and positive orthant of $\mathbb{R}^m$ by $\mathbb{R}^m_+$ and $\mathbb{R}^m_{++}$, respectively. For any $x \in \partial\Omega$, denote the active index set at x by $I(x) = I_0(x) \cup I_1(x)$, where $I_0(x) = \{i \in (1, \ldots, m) : x_i(x) = 0\}$ and $I_1(x) = \{i \in (1, \ldots, m) : x_i(x) = 1\}$.

2. Main Results

To utilize continuation methods, ones generally consider the relaxation of problem (1.1) in the following form:

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} f(x) \\ &\text{s.t.} \quad Ax = b, \\ &\qquad 0 \leq x \leq e. \end{aligned} \tag{2.1}$$

By a homotopy technique, we get the following parameterized unconstrained minimization problem of (2.1):

$$\begin{aligned} \min \quad & (1 - \mu)\left[f(x) - \mu \sum_{i=1}^{n} \alpha_i^{(0)} \ln x_i - \mu \sum_{i=1}^{n} \beta_i^{(0)} \ln(1 - x_i)\right] + \frac{\mu}{2}\|x - x^{(0)}\|^2 \\ \text{s.t.} \quad & Ax = b, \end{aligned} \tag{2.2}$$

where $\alpha_i^{(0)} = y_i^{(0)} x_i^{(0)}$, $\beta_i^{(0)} = z_i^{(0)}(1 - x_i^{(0)})$, $i = 1, \ldots, m$, $x^{(0)} = (x_1^{(0)}, \ldots, x_m^{(0)})^T \in \Omega^0$, and $y^{(0)} = (y_1^{(0)}, \ldots, y_m^{(0)})^T \in \mathbb{R}^m$, $z^{(0)} = (z_1^{(0)}, \ldots, z_m^{(0)})^T \in \mathbb{R}^m$ are constant vectors.

Lemma 2.1. *If* $f(x)$ *is a* C^2 *convex function,* Ω *is nonempty, then for a fixed* $\mu \in (0, 1]$, *then the objective function of* (2.2) *is a strictly convex function.*

Denote $(a, b)^n = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : a < x_i < b, i = 1, 2, \ldots, n\}$, similarly for $(a, b]^n$, $[a, b)^n$, and $[a, b]^n$.

Lemma 2.2. *Let the assumptions in Lemma* 2.1 *hold and the set* $\{x : Ax = b\} \cap (0, 1)^n$ *be nonempty, then for any* $\mu \in (0, 1]$, *problem* (2.2) *has a unique solution.*

Proof. Let

$$U = \left[\frac{1}{6}, \frac{5}{6}\right]^n \bigcap \{x : Ax = b\}.$$

It is easy to show that the objective function of (2.2)

$$\Phi(x, \mu) = (1-\mu)[f(x) - \mu\sum_{i=1}^{n} \alpha_i^{(0)} \ln x_i - \mu\sum_{i=1}^{n} \beta_i^{(0)} \ln(1 - x_i)] + \frac{\mu}{2}\|x - x^{(0)}\|^2,$$

is bounded on U, thus for every $x \in X$, there exists a real number M such that

$$|\psi(x, \mu)| \leq M.$$

When $x_i \to 0$ or $x_i \to 1, \psi(x, \mu) \to \infty$, then there exists a $0 < \sigma < \frac{1}{6}$ such that

$$\Phi(x, \mu) \geq M, \tag{2.3}$$

for all $x \in ((0, \sigma] \cup [1 - \sigma, 1))^n \cap \{x : Ax = b\}$.

Let

$$U_1 = [\sigma, 1 - \sigma]^n \bigcap \{x : Ax = b\}.$$

Since $\Phi(x, \mu)$ is continuous on U_1, and $U \subset U_1$, hence for all $x \in U_1$, there exists $y \in U_1$ such that $\psi(y, \mu) < \psi(x, \mu)$. In addition, for all $x \in (0, 1)^n \setminus U_1$, $\psi(y, \mu) \geq M$. As a result, problem (2.2) has a solution $x^*(\mu) \in (0, 1)^n$. Also, for all $\mu \in (0, 1]$, by Lemma 2.1 and the convexity of the feasible region of problem (2.2), the solution to problem (2.2) is unique. □

To ensure a $0 - 1$ solution, we often add extra penalty terms to the objective function, one way of doing so is to introduce the term

$$\sum_{j \in J} x_j (1 - x_j), \tag{2.4}$$

with a penalty parameter $\gamma > 0$, where J is the index set of the variables that are judged to require forcing to a bound. The problem then becomes

$$\begin{aligned} \min \quad & (1 - \mu)[f(x) - \mu \sum_{i=1}^{n} \alpha_i^{(0)} \ln x_i - \mu \sum_{i=1}^{n} \beta_i^{(0)} \ln(1 - x_i)] \\ & + \frac{\mu}{2} \| x - x^{(0)} \|^2 + \gamma \sum_{j \in J} x_j (1 - x_j) \\ \text{s.t.} \quad & Ax = b. \end{aligned} \tag{2.5}$$

In general, under suitable assumptions, the following two problems have the same minimizers for a sufficiently large γ:

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \ & g(x) \\ \text{s.t.} \quad & c(x) \leq 0, \\ & d(x) = 0, \end{aligned} \tag{2.6}$$

and

$$\min_{x \in \mathbb{R}^n} g(x) + \gamma \sum_{j=1}^{n} x_j (1 - x_j)$$

$$\text{s.t.} \quad c(x) \le 0,$$

$$d(x) = 0, \tag{2.7}$$

which implies the following theorem:

Theorem 2.1. *Suppose* $x(\mu, \gamma)$ *is a local minimizer of* (2.5), *if* $\gamma \to \infty$ *and* $\mu \to 0$, *then for* $j \in J$, $x(\mu, \gamma) = 0$ *or* 1.

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